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INSTABILITY OF EQUILIBRIUM AND APPEARANCE OF ORDERED SPATIAL STRUCTURES ON THE FREE SURFACE OF FERROFLUID

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The process of ordered spatial structures' formation on the free surface of a magnetizable fluid in a steady magnetic field is considered. The stability boundary for the equilibrium state of the ferrofluid layer in a cylindrical solenoid with circular and rectangular cross-sections is defined in a range of physical parameters. The initial stage of the flat free surface evolution is investigated when stability is lost.

Introduction. The formation and the growth of ordered spatial configurations on the free surface of a magnetic fluid (MF) are among the most interesting phenomena accompanying the interaction with the magnetic field. The increase of field induction usually leads to a sequence of bifurcations of the equilibrium state of magnetic fluid and to the exceptional diversity of the free surface. A fairly complete up-to-date picture of the obtained results can be deduced from monographs [1]–[3] as well as from relatively recent works [4]–[10].

In [4], the stability of a flat free surface of ferrofluid is studied. A magnetic field fills completely a vertical cylindrical cavity cut out in a magnet with a flat horizontal surface. It is assumed that the fluid permeability is equal to that of the magnet material. An approximate problem solution for the evolution of initial deviation of the free surface from the horizontal level in a vertical uniform magnetic field is obtained in the linear formulation.

In [5], the evolution of the free surface of a thin layer of ferrofluid, which covers a horizontal plate in a uniform oblique magnetic field is considered. The influence of the plate magnetization on the stability of the flat free surface is investigated there.

In [6]–[10], in the framework of linear theory the stability problem of an infinite horizontal layer of magnetic fluid in a uniform vertical magnetic field is considered. The maximum instability increment and corresponding to it wavenumber dependent on viscosity, the depth of the fluid layer and other physical parameters are determined.

Results of the experimental investigations on the stability of equilibrium structures on the ferrofluid free surface in a normal magnetic field are presented in [11]–[13].

From the review given above we can conclude that the processes of emergence and growth of spatio-temporal structures are usually considered as a manifestation of instability of the initial equilibrium state of magnetic fluid and a transition to a new steady state. It also worth noting that the initial state is mainly taken as an infinite horizontal layer of a uniform magnetic fluid. In a real experiment, the boundedness of the fluid volume and its actual geometry are the main factors determining the process of transition to a new state when stability is lost.

In this paper, the impact of the aforementioned factors is investigated with a sample of the ferrofluid layer, which is located in a cylindrical solenoid. For

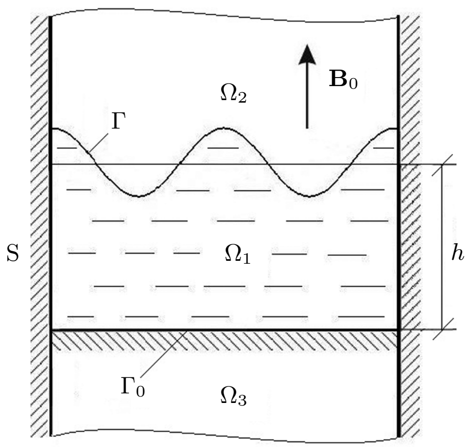

Fig. 1. Sketch of the problem geometry.

cylinders with rectangular and circular cross-sections, boundary regions of stability are built and the most rapidly growing modes determining the initial stages of the evolution of the free surface and structures arising on it are identified.

1. Problem formulation. Let us consider a MF layer located inside an infinite cylinder (solenoid) with a perfectly-conducting wall limited from below by a horizontal solid substrate and having a free surface at the top (Fig. 1). We assume that the fluid is under the action of surface tension forces, steady magnetic and gravitational fields. The region occupied by the fluid in the state of equilibrium is denoted as Ω_1; Ω_2 is the region, which is adjacent to the top of the free surface Γ and filled with a gas; Ω_3 is the semi-infinite region of the cylinder under the surface of contact of the fluid with the plate Γ_0.

The medium in the region Ω_3 is deemed to be homogeneously magnetized. The relationship between the induction $\mathbf{B}$ and the magnetic field strength $\mathbf{H}$ in each of the regions Ω_k, $k = \overline{1,3}$ can be written as

$$\mathbf{B}^{(k)} = \mu_0 \mu_k (H^{(k)}) \mathbf{H}^{(k)} = \mu_0 (\mathbf{H}^{(k)} + \mathbf{M}^{(k)}(H^{(k)})),\ k = \overline{1,3}, \tag{1}$$

where μ_0 is the absolute magnetic permeability of vacuum, $\mu_k(H^{(k)})$ is the relative magnetic permeability of the k-th medium, $\mathbf{M}^{(k)}$ is the magnetization of the k-th medium. The functions $\mu_k : H \to \mu_k(H), k = \overline{1,3}$ are considered as given. It follows that the upper index in parentheses indicates the number of the region, to which the value is pertained.

We assume that the contact angle of the fluid with the side wall of the cylinder S is equal to $\pi/2$; the horizontal free surface $z = 0$ and the vertically directed uniform magnetic field $\mathbf{B} = B_0 \mathbf{e}_z$ ($B_0 = \text{const}$) correspond to the equilibrium state of the fluid. It is well known that with the increasing magnetic field induction the considered equilibrium becomes unstable, the fluid goes into a new equilibrium state with a more sophisticated shape of the free surface. It can be shown that the critical values of the magnetic field induction, the excess of which leads to a loss of stability, do not depend on the fluid viscosity. In what follows, the viscosity of the fluid will be neglected.

In this case, small perturbations of the equilibrium state and the further evolution of the free surface are determined by the potential components of the velocity field of the fluid $\mathbf{v} = \nabla\varphi(t, \mathbf{x})$ ($\varphi(t, \mathbf{x})$ is a potential of the velocity field).

Let $\zeta(t,x,y)$ denote as a deviation of the free surface of the fluid from the horizontal level and $\psi^{(k)}(t,\mathbf{x}), k=\overline{1,3}$ as a perturbation of the magnetic field potential in the region Ω_k. In the linear approximation, the motion of the fluid near the equilibrium state is described by the following system of equations (with respect to φ,ζ,ψ) [14]:

$$\triangle\varphi(t,\mathbf{x})=0 \quad \text{in } \ \Omega_1; \tag{2}$$

$$\frac{\partial\zeta}{\partial t}=\frac{\partial\varphi}{\partial n} \quad \text{on } \Gamma; \tag{3}$$

$$\frac{\partial\varphi}{\partial n}=0 \quad \text{on } S, \tag{4}$$

$$\rho\frac{\partial\varphi}{\partial t}+(-\sigma\triangle_\Gamma\zeta+\rho g\zeta)+B_0\left(\gamma_2\frac{\partial\psi^{(2)}}{\partial z}-\gamma_1\frac{\partial\psi^{(1)}}{\partial z}\right)=c_\Gamma(t) \quad \text{on } \Gamma; \tag{5}$$

$$\frac{\partial\zeta}{\partial n}=0 \quad \text{on } \partial\Gamma; \tag{6}$$

$$\text{div}\mu_k\hat{\nabla}^{(k)}\psi^{(k)}=0, \qquad k=\overline{1,3} \qquad \text{in } \Omega_k, \tag{7}$$

$$\psi^{(2)}-\psi^{(1)}=\frac{B_0(\mu_1-\mu_2)}{\mu_0\mu_1\mu_2}\zeta, \quad \mu_1\gamma_1\frac{\partial\psi^{(1)}}{\partial z}=\mu_2\gamma_2\frac{\partial\psi^{(2)}}{\partial z} \quad \text{on } \Gamma, \tag{8}$$

$$\psi^{(1)}=\psi^{(3)}, \quad \mu_1\gamma_1\frac{\partial\psi^{(1)}}{\partial z}=\mu_3\gamma_3\frac{\partial\psi^{(3)}}{\partial z} \quad \text{on } \Gamma_0, \tag{9}$$

$$\frac{\partial\psi}{\partial n_S}=0, \qquad \text{on } S, \tag{10}$$

$$\psi(t,\mathbf{x})\to 0 \quad \text{at} \quad |\mathbf{x}|\to\infty, \tag{11}$$

$$\zeta(0,x,y)=\zeta_0(x,y), \qquad \frac{\partial\zeta(0,x,y)}{\partial t}=\zeta_0'(x,y) \quad \text{on } \Gamma, \tag{12}$$

$$\hat{\nabla}^{(k)}(\cdot):=\nabla(\cdot)+\frac{\mu_H^{(k)}}{\mu_k}\mathbf{H}_0^{(k)}\frac{\partial(\cdot)}{\partial z},$$

$$\gamma_k:=1+\frac{\mu_H^{(k)}H_0^{(k)}}{\mu_k}, \quad \mu_H^{(k)}:=\left(\frac{\mathrm{d}\mu_k}{\mathrm{d}H}\right)\bigg|_{H=H_0^{(k)}}, \quad k=\overline{1,3}.$$

Here σ is the coefficient of surface tension on Γ; ρ is the fluid density, g is the acceleration of gravity forces, Δ_Γ is the Laplace–Beltrami operator on Γ, $\mathbf{n}$ is the normal to the surface Γ external with respect to the region Ω_1, $\mathbf{n}_S$ is the normal to the surface S, $c_\Gamma(t)$ is an arbitrary time function. The functions $\zeta_0(x,y),\zeta_0'(x,y)$ in Eq. (12) determine the initial deviation and velocity of the fluid free surface.

The function $\zeta(t,x,y)$ must satisfy the condition:

$$\int\limits_\Gamma \zeta(t,x,y)\mathrm{d}\Gamma=0 \qquad \forall t\geq 0. \tag{13}$$

In the considered case, the classical problem of small movements of a capillary ideal fluid [15] is supplemented with the equations and boundary conditions (7)–(11), which determine the perturbation of the magnetic field by perturbations at the free surface Γ. In addition, the dynamic condition (5) on the free surface Γ contains an additional term due to polarization forces of the magnetic field.

2. Evolution of small perturbations. At first, we consider normal eigenfluctuations of fluid, which are described by a solution to the problem (2)– (12) and have the form:

$$(\zeta, \varphi, \psi) = (\zeta(x, y), \varphi(\mathbf{x}), \psi(\mathbf{x})) \exp(i\omega t), \tag{14}$$

where ω is the circular frequency, $\zeta(x, y), \varphi(\mathbf{x}), \psi(\mathbf{x})$ are the modes of the free surface oscillation, velocity potential and the potential of magnetic field perturbation, respectively. Functions (14) can be easily determined by solutions to the spectral boundary value problem:

$$-\triangle u(x, y) = \nu\, u(x, y) \quad \text{in } \Gamma, \quad \frac{\partial u}{\partial n_S} = 0 \quad \text{on } \partial\Gamma. \tag{15}$$

As known, problem (15) has a discrete spectrum $\{\nu_j\}_{j=1}^{\infty}$, which consists of positive eigenvalues ν_j of finite multiplicity, $\nu_j \to \infty$ at $j \to \infty$; the eigenfunctions u_j, which correspond to the eigenvalues ν_j, create a basis $\{u_j\}_{j=1}^{\infty}$ in a Hilbert space $H(\Gamma) := L_2(\Gamma) \ominus \{1\}$. The functions $u_j \in H(\Gamma)$ are orthogonal to the constants in $L_2(\Gamma)$ and may be chosen so as to satisfy the relation:

$$(u_j, u_k)_0 := \int_\Gamma u_j u_k \,\mathrm{d}\Gamma = \delta_{jk}, \quad \forall j, k = 1, 2, 3, ..., \tag{16}$$

where δ_{jk} is the Kronecker symbol.

Let φ_j denote a solution to problem (2)–(4), where instead of (3) one should take the condition:

$$\frac{\partial \varphi_j}{\partial n} = u_j(x, y) \quad \text{on } \Gamma. \tag{17}$$

It is easy to verify that the functions φ_j have the form:

$$\varphi_j = \frac{u_j(x, y)}{\sqrt{\nu_j}\sinh(\sqrt{\nu_j}\,h)}\cosh(\sqrt{\nu_j}(z + h)) \quad (-h \le z \le 0). \tag{18}$$

We introduce the functions $\psi_j := (\psi_j^{(1)}, \psi_j^{(2)}, \psi_j^{(3)})$, which are defined as solutions of problems (7)–(11), and $\zeta = u_j$. The functions ψ_j have the form:

$$\psi_j^{(1)} = -A_j u_j(x, y) \left[\cosh\!\left(\sqrt{\frac{\nu_j}{\gamma_1}}(z + h)\right) + \frac{\mu_3}{\mu_1}\sqrt{\frac{\gamma_3}{\gamma_1}}\sinh\!\left(\sqrt{\frac{\nu_j}{\gamma_1}}(z + h)\right)\right],$$

$$\psi_j^{(2)} = A_j u_j(x, y)\left[\frac{\mu_1}{\mu_2}\sqrt{\frac{\gamma_1}{\gamma_2}}\sinh\!\left(\sqrt{\frac{\nu_j}{\gamma_1}}h\right) + \frac{\mu_3}{\mu_1}\sqrt{\frac{\gamma_3}{\gamma_1}}\cosh\!\left(\sqrt{\frac{\nu_j}{\gamma_1}}h\right)\right]\exp\!\left(-\sqrt{\frac{\nu_j}{\gamma_2}}z\right), \tag{19}$$

$$\psi_j^{(3)} = -A_j u_j(x, y) \exp\left(\sqrt{\frac{\nu_k}{\gamma_3}}(z + h)\right),$$

where

$$A_j := \frac{B_0(\mu_1 - \mu_2)}{\mu_0\Big[\Big(\mu_1^2\sqrt{\frac{\gamma_1}{\gamma_2}} + \mu_2\mu_3\sqrt{\frac{\gamma_3}{\gamma_1}}\Big)\sinh\!\Big(\sqrt{\frac{\nu_j}{\gamma_1}}h\Big) + \mu_1\Big(\mu_2 + \mu_3\sqrt{\frac{\gamma_3}{\gamma_2}}\Big)\cosh\!\Big(\sqrt{\frac{\nu_j}{\gamma_1}}h\Big)\Big]}.$$

For further convenience, we can write the eigenvalues of problem (15) as $\nu_j := (k_j/L)^2$, where L is a characteristic linear dimension of the problem; the oscillation eigenfrequency of fluid determines the dispersion equation (in dimensionless variables):

$$\lambda_j := \frac{\rho L^3 \omega_j^2}{\sigma} = \tanh\left(\frac{k_j h}{L}\right)\left(k_j^3 - k_j^2 \mathrm{W} + k_j \mathrm{Bo}\right), \; j = 1, 2, 3, \ldots, \tag{20}$$

$$k_j := \sqrt{\nu_j}L, \quad \mathrm{Bo} := \frac{\rho g L^2}{\sigma}, \quad \mathrm{W} := \mathrm{KV},$$

$$\mathrm{K} := \frac{\left(\mu_1\sqrt{\gamma_1}\tanh\left(\frac{k_j h}{\sqrt{\gamma_1}L}\right) + \mu_3\sqrt{\gamma_3}\right)(\mu_1\sqrt{\gamma_1} + \mu_2\sqrt{\gamma_2})}{(\mu_1^2\gamma_1 + \mu_2\mu_3\sqrt{\gamma_2\gamma_3})\tanh\left(\frac{k_j h}{\sqrt{\gamma_1}L}\right) + \mu_1\sqrt{\gamma_1}\left(\mu_2\sqrt{\gamma_2} + \mu_3\sqrt{\gamma_3}\right)}.$$

$$\mathrm{V} := \frac{B_0^2(\mu_1 - \mu_2)^2\sqrt{\gamma_1\gamma_2}L}{\mu_0(\mu_1\mu_2)^2(\mu_1\sqrt{\gamma_1} + \mu_2\sqrt{\gamma_2})\sigma}$$

where Bo is the Bond number, V is a dimensionless parameter, which characterizes the ratio of the polarization forces of the magnetic field to the surface tension forces; K is a dimensionless function of the parameters $h/L, \mu_k, \gamma_k, k_j$. Note that $\mathrm{K} = 1$ at $h = \infty$.

We assume that the dimensionless quantities λ_j are arranged in an increasing order. In the general case, the set $\{\lambda_j\}_{j=1}^{\infty}$ has a negative$n$, zero n_0 and a countable set of positive eigenvalues λ_j such that

$$\lambda_j^- := \lambda_j < 0\,\forall j \in \overline{1,n};\ \lambda_j^0 := \lambda_{n+j} = 0\,\forall j \in \overline{1,n_0};\ \lambda_j^+ := \lambda_{n+n_0+j} > 0\,\forall j \in \overline{1,\infty}.$$

The functions $u_j(x,y)$, which correspond to $\lambda_j^{\pm}$, λ_j^0, are denoted as $u_j^{\pm}(x,y)$, $u_j^0(x,y)$. Note that the values λ_j^+ correspond to real eigenfrequencies of the fluid oscillations

$$\omega_j^{\pm} = \pm(\lambda_j^+\sigma/(\rho L^3))^{1/2},\ j \in \overline{1,\infty},$$

the negative values λ_j^- are the growth factors of the perturbations

$$\Lambda_j = |\lambda_j^-\sigma/(\rho L^3)|,\ j \in \overline{1,n}.$$

Let us represent the functions $\zeta_0(\mathbf{x}), \zeta_0'(\mathbf{x})$ in the form of a Fourier series:

$$\zeta_0 = \sum_{j=1}^{\infty}\alpha_j u_j(x,y) = \sum_{j=1}^{n}\alpha_j^- u_j^-(x,y) + \sum_{j=1}^{n_0}\alpha_j^0 u_j^0(x,y) + \sum_{j=1}^{\infty}\alpha^+ u_j^+(x,y),$$

$$\zeta_0' = \sum_{j=1}^{\infty}\beta_j u_j(x,y) = \sum_{j=1}^{n}\beta_j^- u_j^-(x,y) + \sum_{j=1}^{n_0}\beta_j^0 u_j^0(x,y) + \sum_{j=1}^{\infty}\beta^+ u_j^+(x,y),$$

where

$$\alpha_j^{\pm} := (\zeta_0, u_j^{\pm})_0,\ \alpha_j^0 := (\zeta_0, u_j^0)_0,\ \beta_j^{\pm} := (\zeta_0', u_j^{\pm})_0,\ \beta_j^0 := (\zeta_0', u_j^0)_0.$$

The solution to the Cauchy problem (2)–(12) in the accepted notations has the form

$$\zeta(t,x,y) = \sum_{j=1}^{n}\left[\alpha_j^-\cosh\!\left(\Lambda_j^{1/2}t\right) + \frac{\beta_j^-}{\Lambda_j^{1/2}}\sinh\!\left(\Lambda_j^{1/2}t\right)\right]u_j^-(x,y) +$$

$$+\sum_{j=1}^{n_0}\left[\alpha_j^0 + \beta_j^0 t\right]u_j^0(x,y) + \sum_{j=1}^{\infty}\left[\alpha_j^+\cos(|\omega_j^+|t) + \frac{\beta_j^+}{|\omega_j^+|}\sin(|\omega_j^+|t)\right]u_j^+(x,y). \quad (21)$$

The functions $\varphi(t,\mathbf{x})$, $\psi(t,\mathbf{x})$ are not shown here because further they will not be needed.

As follows from Eq. (21), at $n > 0$ the flat free surface becomes unstable, because arbitrary initial perturbations of the surface increase indefinitely with the time t. Unlimited increase of $\zeta(t,x,y)$ occurs because the accepted mathematical model does not take into account the non-linear effects. The above solution (21) gives an indication of the initial stage of the fluid transition process to a new equilibrium state.

The instability condition of the equilibrium flat free surface of magnetizable fluid, as it is easy to derive from Eqs. (20),(21), has the form:

$$\mathrm{W} > \mathrm{W}^* = \min_j \left(k_j + \frac{\mathrm{Bo}}{k_j} \right) \text{ – instability condition.} \tag{22}$$

As a characteristic linear dimension of the solenoid cross-section we choose L. Note that with the changing of L, while maintaining the geometric similarity of the solenoid cross-section, the spectrum $\{k_j\}_{j=1}^{\infty}$ of the dimensionless wavenumbers k_j remains unchanged. We select a sequence of Bond numbers $\mathrm{Bo_n} = k_n^2$ corresponding to the sequence of characteristic lengths $L_n = k_n/\alpha$, where $\alpha := (\rho g/\sigma)^{1/2}$. In this case, the minimum of the right-hand side of Eq. (22) is reached at $k_n = \mathrm{Bo_n^{1/2}}$, so that

$$\mathrm{W}_n^* = 2\mathrm{Bo_n^{1/2}} \quad (\mathrm{Bo_n^{1/2}} = \mathrm{k_n}), \quad \mathrm{n} = 1, 2, 3, \ldots . \tag{23}$$

It follows that $\mathrm{W}^*/\mathrm{Bo}^{1/2} \to 2$ at $\mathrm{Bo} \to \infty$ and, consequently, at $L \to \infty$. The instability condition of the unlimited (in the x and y variables) fluid layer takes the form:

$$\frac{\left(\mu_1\sqrt{\gamma_1}\tanh\left(\alpha h/\sqrt{\gamma_1}\right) + \mu_3\sqrt{\gamma_3}\right)\left(\mu_1\sqrt{\gamma_1} + \mu_2\sqrt{\gamma_2}\right)}{\left(\mu_1^2\gamma_1 + \mu_2\mu_3\sqrt{\gamma_2\gamma_3}\right)\tanh\left(\alpha h/\sqrt{\gamma_1}\right) + \mu_1\sqrt{\gamma_1}\left(\mu_2\sqrt{\gamma_2} + \mu_3\sqrt{\gamma_3}\right)} \times$$
$$\times \frac{B_0^2(\mu_1 - \mu_2)^2\sqrt{\gamma_1\gamma_2}}{\mu_0(\mu_1\mu_2)^2(\mu_1\sqrt{\gamma_1} + \mu_2\sqrt{\gamma_2})\sqrt{\rho g \sigma}} > 2, \quad \alpha := \left(\frac{\rho g}{\sigma}\right)^{1/2}. \tag{24}$$

From Eq. (24), at $h \to \infty$, we derive an instability condition for the MF infinitely deep layer:

$$\frac{B_0^2(\mu_1 - \mu_2)^2\sqrt{\gamma_1\gamma_2}}{\mu_0(\mu_1\mu_2)^2(\mu_1\sqrt{\gamma_1} + \mu_2\sqrt{\gamma_2})} > 2(\rho g \sigma)^{1/2}. \tag{25}$$

In the case $\mu_2 \equiv 1$, this condition can be written as

$$M^{(1)2} > M_{\mathrm{cr}}^2 := \frac{2}{\mu_0}\left(\mu_1 + \frac{1}{\sqrt{\gamma_1}}\right)(\rho g \sigma)^{1/2}, \tag{26}$$

where M_{cr} is the critical value of the fluid magnetization. In such a form, in a slightly different notation, the instability condition for the infinitely deep fluid layer was obtained in [1].

3. Borders of the stability regions. Spatial structures on the free surface. The process of emergence and growth of ordered structures on the MF free surface for a cylinder with the rectangular cross-sectional dimension is considered. The horizontal coordinate axes Ox, Oy direct along the lateral sides of the cylinder. The sectional sizes in the direction of the axes Ox, Oy are denoted as L_x, L_y respectively. As a characteristic linear dimension, we choose $L = S^{1/2} = (L_x L_y)^{1/2}$. This choice makes possible a comparative analysis of the physical parameters' critical values for solenoids with the same cross-sectional area S.

In this case, the modes of the fluid eigenfrequencies u_j and wavenumbers k_j have the form

$$u_j = a_j \cos\frac{m_j \pi x}{L_x} \cos\frac{n_j \pi y}{L_y}, \quad k_j := \pi\left(m_j^2\,\kappa_L + \frac{n_j^2}{\kappa_L}\right)^{1/2}, \tag{27}$$

$$\kappa_L := L_y/L_x, \quad m_j + n_j > 0, \quad m_j, n_j = 0, 1, 2, \dots .$$

The amplitude factors a_j in Eq. (27) are easily determined from the normalization condition (16). The border region of stability in the space of the dimensionless parameters Bo, W, κ_L, $\overline{h} := h/L$ is defined by the equation:

$$\mathrm{W}^*(\mathrm{Bo}, \kappa_L, \overline{h}) = k_j + \frac{\mathrm{Bo}}{k_j} \quad (k_{j-1}k_j \le \mathrm{Bo} \le k_j k_{j+1}),\ j = 1, 2, 3, \dots . \tag{28}$$

In Eq. (28) we assume that all wavenumbers k_j supplemented with a number $k_0 \equiv 0$ are arranged in the increasing order.

On the plane (Bo, W), the stability boundary is a broken line with straight elements. The dependence of W^* on Bo at $\kappa_L = 3/4$, $\overline{h} = \infty$ is presented in Fig. 2. Dotted lines in the instability region $\mathrm{W} > \mathrm{W}^*$ outline zones, within which the most quickly growing modes of the perturbations $u_j(x, y)$ correspond to certain numbers (m_j, n_j). Exactly, these modes provide a visual representation of the structure on the free surface, which appears in the supercritical region of the parameter values $\mathrm{W} > \mathrm{W}^*$. It is assumed here that the parameter W takes supercritical values within a time much less than the characteristic time of hydrodynamic processes.

Note that if we know W^* and the number of modes of the most dangerous perturbations m_j, n_j, we can determine the critical values of the magnetic field induction B^* without any difficulties.

The influence of the parameter κ_L on the critical values W^* is significant for comparatively small values of the Bond number Bo. With increasing Bo, the critical values of W^* for different κ_L practically coincide.

A typical example of the free surface shape, which corresponds to the modes of perturbation (27), is shown in Fig. 2. Our attention is drawn to the fact that at certain values of the parameters κ_L, Bo one-dimensional structures in the form of

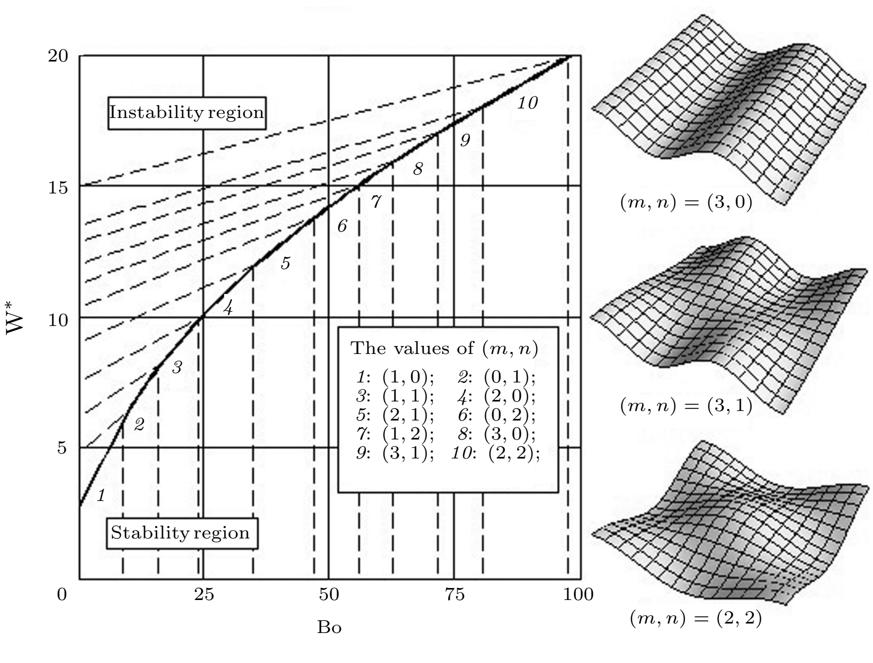

Fig. 2. The boundary region of stability, numbers of the most rapidly growing perturbation modes and shapes of the free surface for a cylinder with the rectangular cross-sectional dimension.

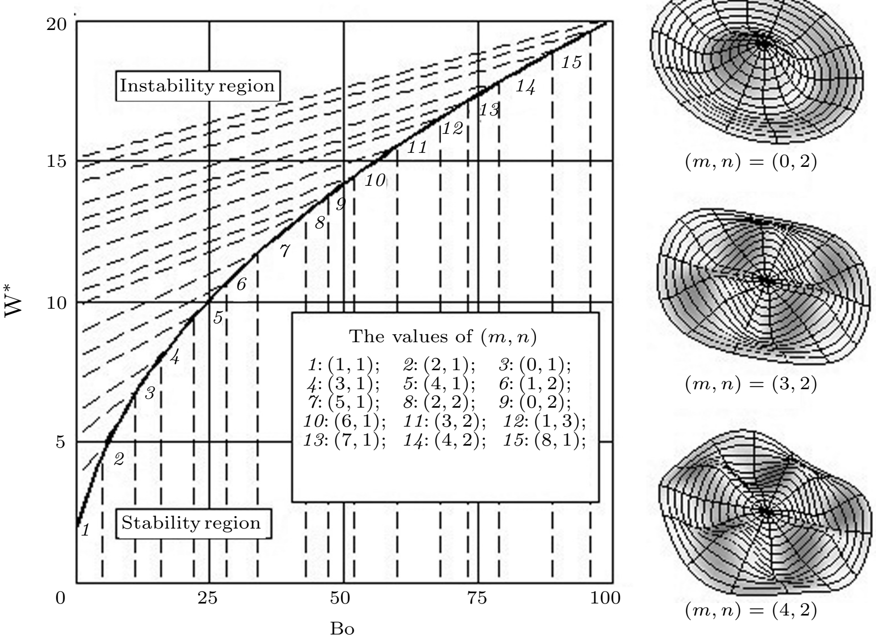

Fig. 3. The boundary region of stability, numbers of the most rapidly growing perturbation modes and shapes of the free surface for a circular cylinder.

corrugated free surface may occur, which correspond to perturbation modes with numbers $m_j > 0$, $n_j = 0$ or $m_j = 0$, $n_j > 0$.

For the circular cylinder, the stability condition has the same form (22) as for a cylinder with the rectangular cross-sectional dimension. In this case, the perturbation modes in a cylindrical coordinate system r, ϑ, z have the form:

$$u_j = a_j J_{m_j}(k_{m_j n_j} r/L) \left\{ \begin{array}{l} \cos(m_j\vartheta), \\ \sin(m_j\vartheta), \end{array} \right. \quad m_j = 0, 1, 2, \ldots, \; n_j = 1, 2, \ldots. \tag{29}$$

There $J_m(\cdot)$ is the Bessel function of the first kind of order m, k_{mn} is the n-th non-zero root of the equation $J'_m(k) = 0$, R is the radius of the cylinder. In Eqs. (20), (22), we accept that $L = R$, $k_j := \kappa_{m_j n_j}$. The calculation results of the boundary region of stability and numbers of the most rapidly growing perturbation modes at $h = \infty$ for the circular cylinder are illustrated in Fig. 3. Two-periodic forms of the most rapidly growing perturbation modes for this case are in qualitative agreement with the results obtained in [11, 12].

For fixed values of Bo, the growth of $\mathrm{W}(> \mathrm{W}^*)$ is accompanied by an increase of the integer parameters m_j, n_j and wavenumbers k_j corresponding to the most rapidly growing perturbation. It leads to a decrease of the internal scale of the structures (the distance between adjacent peaks and troughs), emerging on the free surface of the fluid in the supercritical region of parameter values.

4. Conclusions.

• The stability boundaries for the equilibrium states of a flat layer of magnetic fluid were constructed for the cases of a circular cylinder and a cylinder with the rectangular cross-sectional dimension. The most rapidly growing perturbation modes, describing the initial stage evolution of the free surface when stability is lost, are determined. In contrast to the case of infinitely extended horizontal layer of ferrofluid, eigenfrequencies of finite volume fluid oscillation form a discrete spectrum and the corresponding modes of oscillation depend on the geometry of the side wall of the solenoid. The influence of the side wall is particularly noticeable at low values of the Bond number.

• The range of the physical parameter values, corresponding to unstable equilibrium states of a flat MF layer, is divided into zones, each of which is characterized by a well-defined mode of the most rapidly growing perturbation. The changes of the magnetic field induction, causing the transition of physical parameters from one zone to another, are accompanied by a qualitative reconstruction of the free surface. This effect has been reported previously in [11, 12].

• With increasing of the Bond number, the influence of the solenoid sidewall becomes weaker. The critical values of the magnetic field induction B^* tend to values corresponding to the ferrofluiud infinite layer.

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