On the structure of quantum channels

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Let H and K be finite dimensional Hilbert spaces. In the literature of quantum information theory, a quantum channel from $\mathfrak{B}(H)$ to $\mathfrak{B}(K)$ is described as a linear map

$$\psi : \mathfrak{B}(H)' \to \mathfrak{B}(K)' \tag{1}$$

from the dual of $\mathfrak{B}(H)$ to the dual of $\mathfrak{B}(K)$ which holds property: linear map $\phi : \mathfrak{B}(H) \to \mathfrak{B}(K)$ defined by an equation $\psi = \phi'$ (ψ is the adjoint map to the map ϕ) is a unital completely positive map.

Our aim is to describe the structure of a quantum channel from $\mathfrak{B}(H)$ to $\mathfrak{B}(K)$.

Suppose $\rho \in \mathfrak{B}(H)'$ and a is an arbitrary element of $\mathfrak{B}(H)$; then define $\delta\rho \in \mathfrak{B}(H)$ by the equation

$$\rho(a) = \mathrm{Tr}(a \cdot \delta\rho) \tag{2}$$

Vice versa suppose $s \in \mathfrak{B}(H)$ and a is an arbitrary element of $\mathfrak{B}(H)$; then define $\sum[s] \in \mathfrak{B}(H)'$ by equation

$$\sum[s](a) = \mathrm{Tr}(a \cdot s) \tag{3}$$

Proposition 1. *Let $\psi : \mathfrak{B}(H)' \to \mathfrak{B}(K)'$ be a linear operator. Then*

$$\psi(\rho) = \sum_{m,n=1}^{\dim(H)} \mathrm{Tr}((\delta\rho)e_{mn}) \sum[z_{nm}(\psi)]$$

where

1) $\{e_{mn} \mid m, n = 1, \ldots, \dim(H)\}$ is a family of matrix units in $\mathfrak{B}(H)$,

2) $z_{nm}(\psi) = \delta(\psi(\sum[e_{mn}]))$ for $m, n = 1, \ldots, \dim(H)$.

Theorem 1. *Let $\psi : \mathfrak{B}(H)' \to \mathfrak{B}(K)'$ be a linear operator; then ψ is a quantum channel iff next two conditions are held*

1) suppose $\{a_n \mid n = 1, \ldots, \dim(H)\}$ be a family of elements from $\mathfrak{B}(K)$; then

$$\sum_{m,n=1}^{\dim(H)} \mathrm{Tr}(a_m z_{nm}(\psi) a_n^*) \geq 0;$$

2) for all $m, n = 1, \ldots, \dim(H)$ equality $\mathrm{Tr}(z_{mn}(\psi)) = \delta_{mn}$ is held.

Corollary 1. *Let* $\psi : \mathfrak{B}(H)' \to \mathfrak{B}(K)'$ *be a linear operator,* $Z \in \mathfrak{B}(H) \otimes \mathfrak{B}(K)$ *and* $Z = \sum_{m,n=1}^{\dim(H)} e_{mn} \otimes z_{nm}(\psi)$. ψ *is a quantum channel if and only if* $Z \geq 0$ *and* $\mathrm{Tr}_K(Z) = 1$, *where* $\mathrm{Tr}_K(a \otimes b) = \mathrm{Tr}(b)a$ *for all* $a \in \mathfrak{B}(H)$ *and* $b \in \mathfrak{B}(K)$.

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