The isomorphism problem for finitary incidence rings

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The notion of a finitary incidence algebra was first introduced in [1] as a generalization of the notion of an incidence algebra for the case of an arbitrary poset. It was shown that the isomorphism problem for such algebras was solved positively ([1], Theorem 5). In the present talk we consider this problem in more general case.

Let \( C \) be a preadditive small category. Assume that the binary relation \( \leq \) on the set of its objects, such that \( x \leq y \iff \text{Mor}(x, y) \neq \emptyset \), is a partial order. Consider the set of formal sums of the form

\[
\alpha = \sum_{x \leq y} \alpha_{xy}[x, y],
\]

where \( \alpha_{xy} \in \text{Mor}(x, y) \), \([x, y]\) is a segment of the partial order. A formal sum (1) is called a finitary series, if for any \( x, y \in \text{Ob}C \), \( x < y \) there exists only a finite number of \([u, v] \subset [x, y]\), such that \( u < v \) and \( \alpha_{uv} \neq 0 \). The set of the finitary series is denoted by \( FI(C) \).

The addition of the finitary series is inherited from the addition of the morphisms. The multiplication is defined by means of the convolution:

\[
\alpha \beta = \sum_{x \leq y} \left( \sum_{x \leq z \leq y} \alpha_{xz} \alpha_{zy} \right) [x, y],
\]

where \( \alpha_{xz} \alpha_{zy} = \alpha_{zy} \circ \alpha_{xz} \in \text{Mor}(x, y) \). Under these operations \( FI(C) \) form an associative ring with identity, which is called a finitary incidence ring of a category.

It is easy to see, that the description of the idempotents of \( FI(C) \) can be obtained as in [1]. This allows us to solve the isomorphism problem for finitary incidence rings of preorders.

Let \( R \) be an associative ring with identity, \( P(\preceq) \) an arbitrary preordered set. Denote by \( \sim \) the equivalence relation on \( P \), such that \( x \sim y \iff x \preceq y \) and \( y \preceq x \). Define \( M([x], [y]) \) to be an abelian group of row and column finite matrices over \( R \), indexed by the elements of the equivalence classes \( [x] \) and \( [y] \). Consider the following preadditive category \( C \):

1. \( \text{Ob}C = P/\sim \) with the induced order \( \leq \);

2. For any pair \([x], [y] \in \text{Ob}C \) define \( \text{Mor}([x], [y]) = M([x], [y]) \), if \([x] \leq [y] \), and 0 otherwise (the composition of the morphisms is the matrix multiplication).

Denote the finitary incidence ring of this category by \( FI(P, R) \). Obviously, \( FI(P, R) \) is an algebra over \( R \), which is called a finitary incidence algebra of \( P \) over \( R \).
Theorem 1. Let $P$ and $Q$ be preordered sets, $R$ and $S$ indecomposable commutative rings with identity, $C$ and $D$ preadditive categories corresponding to the pairs $(P, R)$ and $(Q, S)$, respectively. If $FI(P, R) \cong FI(Q, S)$ as rings, then $C \cong D$.

Corollary 1. Let $P$ and $Q$ be class finite preordered sets, $R$ and $S$ indecomposable commutative rings with identity. If $FI(P, R) \cong FI(Q, S)$ as rings, then $P \cong Q$ and $R \cong S$.

As a corollary we obtain the positive solution of the isomorphism problem for weak incidence algebras given in [2].

References


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